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We show how entanglement can be used, without being consumed, to accomplish unitary operations that could not be performed with out it. When applied to infinitesimal transformations our method makes equivalent, in the sense of Hamiltonian simulation, a whole class of otherwise inequivalent two-qubit interactions. The new catalysis effect also implies the asymptotic equivalence of all such interactions.

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Can entanglement help to perform certain tasks? How much entanglement has to be consumed? Can we use entanglement without consuming it at all? These questions are quite relevant in the context of quantum information theory, since entanglement can be considered as an expensive physical resource without classical analogy. In particular, the last question has been recently answered [1] in the context of transformation between states of two parties, Alice and Bob, under local operations and classical communication (LOCC). More specifically, examples have been presented where a state can only be transformed into some other one by LOCC when a certain entangled state $|\eta\rangle_{ab}$ is available. In this case, even though the total entanglement (shared by Alice and Bob) decreases, the state $|\eta\rangle_{ab}$ is recovered after the procedure. This effect has been termed catalysis [1], since the state $|\eta\rangle_{ab}$ is necessary for the process to occur, even though it is not consumed.

In this Letter we present a novel catalysis effect through quantum entanglement. A maximally entangled state will be used, but not consumed, to perform a non-local task that cannot be achieved without it. The task consists of implementing a certain two–qubit unitary gate when only some other one is available. Remarkably, this catalysis is achieved using only local unitary manipulations. The same construction allows to simulate with a given non–local interaction other kinds of interactions, which otherwise could not be simulated using only LOCC. In our method unitarity of the local manipulations is an important feature, since it makes possible that some LOCC-inequivalent interactions become fully equivalent in presence of entanglement. This sharply contrasts with the case of entangled state conversions through LOCC manipulations [1], where LOCC-inequivalent states must remain inequivalent through catalysis, because the local measurements needed in the conversions unavoidably decrease the entanglement between the parties. Another consequence of our results is that certain Hamiltonians become equivalent under asymptotic LOCC, a phenomenon that shares analogies with the one that occurs in transformations between pure states [2].

Let us consider two parties, Alice and Bob, each of them possessing a qubit, A and B, respectively. Their goal is to apply certain unitary operator $\tilde{U}$ to the qubits. However, they only have at hand another particular two–qubit unitary operator U, and the ability to perform one of the following classes of operations. (a) LU: local unitary operations on each qubit; (b) LU+anc: each of the local unitary operations is jointly performed on a local ancilla, initially in a product state, and a qubit; (c) LO: each party can perform general local operations on its qubit (and ancilla); (d) LOCC: the same as LO but classical communication is also allowed; (e) cat–LU: the same as LU+anc, but now Alice's and Bob's ancillas are initially in an entangled state, which can be used, but not consumed, during the process. Clearly, everything that can be done in the LU, LU+anc, and LO scenarios, can be also done in the LOCC scenario. Here we will show that there are operators $\tilde{U}$ that cannot be applied in the LOCC scenario, but that can be achieved in the cat–LU one.

Let U denote a unitary operator acting on two qubits A and B. Using the results of Ref. [3], we can always write $U^{AB} = (u^A v^B)[U_s^{AB}(c_1, c_2, c_3)](\tilde{u}^A \tilde{v}^B)$, where

$$U_s^{AB}(c_1, c_2, c_3) = e^{-i\sum_{k=1}^{3} c_k \sigma_k^A \sigma_k^B}, \tag{1a}$$

$$\pi/4 \geq c_1 \geq c_2 \geq |c_3|, \tag{1b}$$

the σ's are Pauli operators, and the u's and v's are local unitary operators. The superscripts accompanying each operator indicate the system(s) on which it acts. The coefficients c can be easily determined using the method described in Ref. [3]. Any two unitary operators are equivalent under LU (i.e. they can perform the same tasks if arbitrary local unitary operations on A and B are allowed before and after their action) if and only if they give rise to the same $U_s(c_1, c_2, c_3)$. Since in all what follows we will always allow for LU, we can restrict ourselves to unitary operators U of the form (1a).

In the catalytic scenario, cat–LU, we have at our disposal two ancillas (qubits) a and b, initially in the Bell state $|B_{0,0}\rangle_{ab}$ [4]. We must impose that after the whole process the ancillas a and b end up again in state $|B_{0,0}\rangle_{ab}$. We allow for joint unitaries acting on A and a, as well as joint unitaries acting on B and b. We will show that in this situation we can use $U_s(c_1, c_2, c_3)$ to implement $U_s(c_1 + c_2, 0, 0)$. Later on we will show that this cannot be achieved without the entangled ancillas, even if LOCC are allowed.

The above claim about what can be done with U_s in the cat–LU scenario follows directly from the fact that

$$\left(w^{Aa}w^{Bb}\right)^\dagger \left[U_s^{AB}(c_1,c_2,c_3)\right] \left(w^{Aa}w^{Bb}\right)|\Psi\rangle_{AB}|B_{0,0}\rangle_{ab} \\ = e^{ic_3}\left[U_s^{AB}(c_1+c_2,0,0)\right]|\Psi\rangle_{AB}|B_{0,0}\rangle_{ab}, \tag{2}$$

for all $|\Psi\rangle$. Here, the unitary operators w are defined according to $w|i,j\rangle = |j, i\oplus j\rangle$, and therefore correspond to a swap operation followed by a c–NOT. Even though Eq. (2) can be directly checked, we will indicate here the main idea behind this equation. The operators in the form U_s are diagonal in the Bell basis [4], i.e.

$$U_s(c_1,c_2,c_3)|B_{0,0}\rangle = e^{-i(c_1+c_2-c_3)}|B_{0,0}\rangle, \tag{3a}$$
$$U_s(c_1,c_2,c_3)|B_{1,0}\rangle = e^{-i(c_1-c_2+c_3)}|B_{1,0}\rangle, \tag{3b}$$
$$U_s(c_1,c_2,c_3)|B_{0,1}\rangle = e^{i(c_1+c_2+c_3)}|B_{0,1}\rangle. \tag{3c}$$
$$U_s(c_1,c_2,c_3)|B_{1,1}\rangle = e^{-i(-c_1+c_2+c_3)}|B_{1,1}\rangle. \tag{3d}$$

In particular,

$$e^{ic_3}U_s(c_1+c_2,0,0)|B_{\alpha,0}\rangle = e^{-i(c_1+c_2-c_3)}|B_{\alpha,0}\rangle, \tag{4a}$$
$$e^{ic_3}U_s(c_1+c_2,0,0)|B_{\alpha,1}\rangle = e^{i(c_1+c_2+c_3)}|B_{\alpha,1}\rangle, \tag{4b}$$

for $\alpha = \pm 1$. Thus, we see that if we could transform $|B_{\alpha,\beta}\rangle_{AB} \rightarrow |B_{0,\beta}\rangle_{AB}$ before acting with $U_s(c_1,c_2,c_3)$ and then we would invert such transformation, we would obtain the desired result. Unfortunately, there exist no such a transformation since two states ($\alpha = 0,1$) have to be mapped onto a single one, and then back. However, this can be accomplished with the help of the entangled ancillas, and this is precisely what the operator $w_{Aa}w_{Bb}$ does: it transforms $|B_{\alpha,\beta}\rangle_{AB}|B_{0,0}\rangle_{ab} \rightarrow |B_{0,\beta}\rangle_{AB}|B_{\overline{\alpha},\beta}\rangle_{ab}$.

Now, let us show that $U_s(c_1+c_2,0,0)$ cannot be obtained with the help of $U_s(c_1,c_2,c_3)$ and LOCC for a range of values of the parameters c. Note that this automatically implies that this task is not possible either with LU, LU+anc, or LO. In the LOCC scenario we may use two ancillas a and b, with corresponding Hilbert spaces of arbitrary dimensions. The LOCC consist of generalized measurement on A and a, and on B and b involving classical communication before and also after the application of $U_s(c_1,c_2,c_3)$.

We want that the whole procedure involving a set of LOCC, followed by the action of $U_s(c_1,c_2,c_3)$, and again another set of LOCC, reproduce the action of $U_s(c_1+c_2,0,0)$ on any input state of A and B. In particular, we can take A and B initially entangled with two other, remote qubits C and D, in state

$$|\Psi_0\rangle_{ABCDab} \equiv |B_{0,0}\rangle_{AC}|B_{0,0}\rangle_{BD}|0\rangle_a|0\rangle_b. \tag{5}$$

Let us assume that a set of LOCC takes place *before* $U_s(c_1,c_2,c_3)$ acts. We will now show that one can substitute these LOCC by local unitaries acting on A and a, and B and b. We will use the fact that the whole process must be described by a unitary operator $[U_s(c_1+c_2,0,0)]$ acting on A and B, which implies that the entanglement between the qubit C (D) and the rest of the systems must be preserved, i.e. the final state must be a maximally entangled state between C (D) and the rest. For a set of outcomes Γ of the generalized measurements performed on A and a, and on B and b, before the application of $U_s(c_1,c_2,c_3)$ we will have that the state of the systems will change according to $x_\Gamma^{Aa}y_\Gamma^{Bb}|\Psi_0\rangle_{ABCDab}$, where x_Γ and y_Γ are two operators that depend on the set of outcomes of the measurements. Let us consider first the action of x (we will omit the subscript Γ in order to keep the notation readable)

$$x|0\rangle_A|0\rangle_a = d_0|\psi_0\rangle_{Aa}, \quad x|1\rangle_A|0\rangle_a = d_1|\psi_1\rangle_{Aa}, \tag{6}$$

where $|\psi_{0,1}\rangle$ are normalized states. Note that it can occur neither that $|d_0| \neq |d_1|$ nor that $|\psi_0\rangle$ and $|\psi_1\rangle$ are not orthonormal. If this were the case, then the entanglement of the qubit C with the rest of the systems would decrease. According to well known results on entanglement concentration [5], this entanglement cannot be recovered later on with the help of LOCC. Since the whole protocol does not involve joint actions with remote qubit C, this immediately would contradict the fact that this entanglement has to be maintained at the very end of the process. Thus, we must have that $|d_0| = |d_1| \equiv d$ and, at the same time, $|\psi_0\rangle$ and $|\psi_1\rangle$ are orthonormal. But in this case we can always find certain unitary operator u acting on A and a such that du gives the same action as x on the relevant states. Thus, we can substitute x_Γ by a unitary operator u_Γ chosen randomly with probability $|d_\Gamma|^2$. The same analysis applies to y_Γ.

According to this result, the problem reduces to showing that

$$|\Phi_1(\Psi)\rangle \equiv \left[U_s^{AB}(c_1+c_2,0,0)\right]|\Psi\rangle_{AB}|0,0\rangle_{ab}, \tag{7}$$

cannot be obtained starting from

$$|\Phi_2(\Psi)\rangle \equiv \left[U_s^{AB}(c_1,c_2,c_3)\right]\left(x^{Aa}y^{Bb}\right)|\Psi\rangle_{AB}|0,0\rangle_{ab}, \tag{8}$$

using LOCC, for all $|\Psi\rangle$ and where x and y are unitary. In order to prove that, we restrict the values of the parameter c to satisfy $c_3 = 0$, $c_2 > 0$, and $c_1 + c_2 \leq \pi/4$, and use the following fact [6]: if $|\Psi_1\rangle$ can be obtained by LOCC out of $|\Psi_2\rangle$, then

$$P(\Psi_1) \geq P(\Psi_2), \tag{9}$$

where

$$P(\Psi) \equiv \max_{||\psi||=||\phi||=1} |\langle\psi|\langle\phi||\Psi\rangle|^2. \tag{10}$$

[P is the square of the maximal Schmidt coefficient.] In particular, if we take in (7) $|\Psi_{i,j}\rangle_{AB} = |i\rangle_A|j\rangle_B$ ($i,j = 0,1$), we have that $P[\Phi_1(\Psi_{i,j})] = \cos^2(c_1+c_2)$. Defining

$$|\psi_i\rangle_{Aa} \equiv x^{Aa}|i,0\rangle_{Aa}, \tag{11a}$$
$$|\varphi_j\rangle_{Bb} \equiv y^{Bb}|j,0\rangle_{Bb}, \tag{11b}$$

we will show that it is not possible to have

$$|\langle\psi_i|\langle\varphi_j||\Phi_2(\Psi_{i,j})\rangle|^2 \leq \cos^2(c_1+c_2), \tag{12}$$

for all $i,j=0,1$, and therefore that condition (9) is violated. We can always write

$$|\psi\rangle_{Aa}|\varphi\rangle_{Bb} = \sum_{\alpha,\beta=0,1} |B_{\alpha,\beta}\rangle_{AB}|N_{\alpha,\beta}\rangle_{ab}, \tag{13}$$

where the $n_{\alpha,\beta} \equiv ||N_{\alpha,\beta}||^2 \geq 0$ add up to one. Thus, condition (12) reduces to

$$\Big|e^{-i(c_1+c_2)}n_{0,0}+e^{i(c_1+c_2)}n_{0,1} \\ +e^{-i(c_1-c_2)}n_{1,0}+e^{i(c_1-c_2)}n_{1,1}\Big|^2 \leq \cos^2(c_1+c_2). \tag{14}$$

Actually, it can be easily shown that the left hand side is always larger or equal than the right hand side, the equality holding only for $n_{1,0}=n_{1,1}=0$ and $n_{0,0}=n_{0,1}=1/2$. Using these results in Eq. (13) and imposing that $|\psi_i\rangle_{Aa}|\varphi_j\rangle_{Bb}$ is a product state, we obtain that it must be of either of the form $|0,1\rangle_{AB}|\mu_i,\nu_j\rangle_{ab}$ or $|1,0\rangle_{AB}|\mu_i,\nu_j\rangle_{ab}$. Now, recalling that $|\psi_i\rangle_{Aa}|\varphi_j\rangle_{Bb}$ must be created using local unitary operators acting on A and a, and B and b out of $|i,0\rangle_{Aa}|j,0\rangle_{Bb}$ one readily finds that this is impossible for all $i,j=0,1$. Thus, we have proven that $U_s(c_1+c_2,0,0)$ cannot be obtained with the help of $U_s(c_1,c_2,0)$ and LOCC for $\pi/4 \geq c_1+c_2>0$ and $c_1 \geq c_2 > 0$.

In the following, we will analyze the implications of our catalytic method in the context of infinitesimal transformations of two-qubits [7–11]. Remarkably, the study of this kind of transformations has allowed to establish a partial order in the set of all possible physical interactions (or Hamiltonians) [10]. This partial order is related to whether a given interaction can *simulate* (i.e., produce the same results of) another one, when certain operations are allowed. In this context, the necessary and sufficient conditions for a two-qubit Hamiltonian H to be able to simulate another H' under LU, LU+anc and LOCC have been derived [10,11], giving the same conditions. One can immediately see from our general results on unitary operators that in the catalytic scenario, these conditions are relaxed, i.e. there are certain Hamiltonians that can simulate other under cat–LU, but not under LOCC. Here we will analyze this fact in detail and extract some conclusions.

Thus, we consider $U=e^{-iH\delta t}$, where $H=H^\dagger$ is a Hamiltonian acting on the qubits A and B and $||H\delta t|| \ll 1$. Again, since we allow for arbitrary local unitaries at any time, we can restrict ourselves to Hamiltonians of the form

$$H(c_1,c_2,c_3) = \sum_{k=1}^{3} c_k \sigma_k^A \sigma_k^B, \tag{15a}$$

$$c_1 \geq c_2 \geq |c_3|. \tag{15b}$$

In Refs. [10,11] it has been shown that given $H(c_1,c_2,c_3)$, a total time δt, and if we allow for LOCC after time steps smaller than δt, then we can obtain the operation generated by $H(\tilde c_1,\tilde c_2,\tilde c_3)$ during the same time δt up to second order corrections in $H\delta t$ if and only if

$$c_1+c_2-c_3 \geq \tilde c_1+\tilde c_2-\tilde c_3, \tag{16a}$$

$$c_1 \geq \tilde c_1, \tag{16b}$$

$$c_1+c_2+c_3 \geq c_1+c_2+c_3. \tag{16c}$$

This implies that under LOCC, H can simulate $\tilde H$ if and only if these conditions are satisfied.

If we use our catalytic method, we have that it is possible to simulate $\tilde H(c_1+c_2,0,0)$ with $H(c_1,c_2,c_3)$, which for $c_2 \neq 0$ violates condition (16b). In fact, taking $c_3=0$, we see that $H_1 \equiv H(c_1+c_2,0,0)$ can simulate $H_2 \equiv H(c_1,c_2,0)$ as well, since conditions (16) are fulfilled. Thus, our catalytic method makes any pair of Hamiltonians of the form H_1 and H_2 equivalent, although they are inequivalent under LOCC simulation. This result also has fundamental implications in the study of *asymptotic* simulation of interactions using LU+anc. There N applications of an evolution generated by H for a time δt are available, in the limit $\delta t \to 0$ and $N\delta t \to \infty$. H_1 can simulate H_2 even for finite N [10,11]. We can now use H_2 for N_0 times to create a maximally entangled state of the ancillas [7] with $N_0\delta t$ finite, which could then be used to catalyze the Hamiltonian evolution generated by H_1 a number $N-N_0 \sim N$ of times.

So far, we have seen that under the catalytic scenario, some Hamiltonians acting on two qubits become equivalent. Of course, an important question is whether all Hamiltonians become equivalent in that scenario [12]. We now show that this is not the case. We derive a set of necessary conditions similar to (16) that the Hamiltonians H and $\tilde H$ must fulfill for H to be able to simulate $\tilde H$. First, we will use that both Hamiltonians are diagonal in the Bell basis [4], and we will call the corresponding eigenvalues

$$\lambda_1 = c_1+c_2-c_3, \quad \tilde\lambda_1 = \tilde c_1+\tilde c_2-\tilde c_3+\tilde c_4, \tag{17a}$$

$$\lambda_2 = c_1-c_2+c_3, \quad \tilde\lambda_2 = \tilde c_1-\tilde c_2+\tilde c_3+\tilde c_4, \tag{17b}$$

$$\lambda_3 = -c_1+c_2+c_3, \quad \tilde\lambda_3 = -\tilde c_1+\tilde c_2+\tilde c_3+\tilde c_4, \tag{17c}$$

$$\lambda_4 = -c_1-c_2-c_3, \quad \tilde\lambda_4 = -\tilde c_1-\tilde c_2-\tilde c_3+\tilde c_4. \tag{17d}$$

Note that with these numeration, the λ's and $\tilde\lambda$'s are sorted in decreasing order. We have also taken into account a global constant $\tilde c_4$, since it will be important in the discussion below. We will show that if H can simulate $\tilde H$ under cat–LU, then

$$c_1+c_2-c_3 \geq \tilde c_1+\tilde c_2-\tilde c_3+\tilde c_4, \tag{18a}$$

$$c_1+c_2+c_3 \geq \tilde c_1+\tilde c_2+\tilde c_3-\tilde c_4, \tag{18b}$$

$$\sum_{k=1}^{3}|c_k| \geq \sum_{k=1}^{4}|\tilde c_k|. \tag{18c}$$

These conditions mean, for example, that with $H(c_1,c_2,c_3)$ it is not possible to efficiently simulate either $\tilde{H}(c_1+c_2+c_3,0,0)$ —which would imply catalytic equivalence of all interactions since the converse simulation is possible [cf. (16)]—, nor $H(c_1,c_2,-c_3)$ —which excludes the simulation of a time-reversed evolution of $H(c_1,c_2,c_3)$.

Following the same steps as in [11] we find that H can efficiently simulate $\tilde{H}$ using LOCC only if there exists a set of unitary operators u_m and v_m and some positive numbers p_m which add up to one, such that

$$\sum_m p_m (u_m^{Aa} v_m^{Bb})^\dagger \left[H^{AB}\otimes \mathbb{1}^{ab}\right](u_m^{Aa}v_m^{Bb})|\Psi\rangle_{AB}|\Phi_0\rangle_{ab} = \left[\tilde{H}^{AB}\otimes \mathbb{1}^{ab}\right]|\Psi\rangle_{AB}|\Phi_0\rangle_{ab}, \tag{19}$$

for all $|\Psi\rangle$ and certain fixed state $|\Phi_0\rangle$ of arbitrary dimensional ancillas. Here we have included $\mathbb{1}^{ab}$ to make the formula more explicit. According to a basic result in the theory of majorization [13], the operator resulting from the sum over m in (19) must have the eigenvalues lying in the interval $[\lambda_1,\lambda_4]$. This automatically implies that the operator $\tilde{H}^{AB}$ must also have its eigenvalues in the same interval, which leads to $\lambda_1\geq\tilde{\lambda}_1$ and $\lambda_4\leq\tilde{\lambda}_4$, and therefore to (18a,18b). In order to obtain the last condition (18c), we apply the bra ${}_{ab}\langle\Phi_0|$ to both sides of Eq. (19), multiply the corresponding equation by $\sigma_k^A\sigma_k^B/4$ and trace with respect to A and B. Taking the absolute values of the resulting expressions, and adding from $k=1,\ldots,4$ we obtain

$$\sum_{k=1}^{4}|\tilde{c}_k| \leq \sum_{n=1}^{3}|c_n|\sum_m p_m h_{n,m}, \tag{20}$$

where

$$h_{n,m}=\frac{1}{4}\sum_{k=1}^{4}|{}_{ab}\langle\Phi_0|X^a_{k,n,m}Y^b_{k,n,m}|\Phi_0\rangle_{ab}|, \tag{21}$$

and

$$X^a_{k,n,m}=\mathrm{tr}_A[\sigma_k^A(u_m^{Aa})^\dagger\sigma_m^A u_m^{Aa}]. \tag{22a}$$

$$Y^b_{k,n,m}=\mathrm{tr}_B[\sigma_k^B(v_m^{Bb})^\dagger\sigma_m^B v_m^{Bb}]. \tag{22b}$$

Using Cauchy–Schwarz inequality, we have

$$h_{n,m}\leq\left[\frac{1}{4}\sum_{k=1}^{4}\langle\Phi_0|X^a_{k,n,m}(X^a_{k,n,m})^\dagger|\Phi_0\rangle\right]^{1/2} \times\left[\frac{1}{4}\sum_{k=1}^{4}\langle\Phi_0|Y^b_{k,n,m}(Y^b_{k,n,m})^\dagger|\Phi_0\rangle\right]^{1/2}=1,$$

where for the last equality we have used the fact that σ_k form an orthonormal basis in the space of operators acting on a qubit. Substituting $h_{n,m}\leq 1$ in Eq. (20), we finally obtain condition (18c).

In conclusion, we have shown that certain unitary operations can be catalyzed by an entangled state, in the sense that the state is not consumed but without it the process would not be possible. We have also shown that the method introduced here allows to make equivalent certain kind of interactions acting on two qubits. This fact allows for these interactions to become equivalent in the asymptotic limit, which is compatible with the conjecture that all two-qubit interactions are equivalent in the asymptotic limit [12].

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$$|B_{0,0}\rangle\equiv\frac{1}{\sqrt{2}}(|0,1\rangle+|1,0\rangle),\ |B_{1,0}\rangle\equiv\frac{1}{\sqrt{2}}(|0,0\rangle+|1,1\rangle),$$
$$|B_{0,1}\rangle\equiv\frac{1}{\sqrt{2}}(|0,1\rangle-|1,0\rangle),\ |B_{1,1}\rangle\equiv\frac{1}{\sqrt{2}}(|0,0\rangle-|1,1\rangle).$$

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